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Finite Blaschke product interpolation on the closed unit disc

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Abstract

We show how to construct all finite Blaschke product solutions and the minimal scaled Blaschke product solution to the Nevanlinna–Pick interpolation problem in the open unit disc by solving eigenvalue problems of the interpolation data. Based on a result of Jones and Ruscheweyh we note that there always exists a finite Blaschke product of degree at most $n-1$ that maps n distinct points in the closed unit disc, of which at least one is on the unit circle, into n arbitrary points in the closed unit disc, provided that the points inside the unit circle form a positive semi-definite Pick matrix of full rank. Finally, we discuss a numerical limiting procedure.

Keywords: Interpolation; Finite Blaschke product

1. Introduction

Our terminology follows the book of Garnett [3]. Let H^∞ denote the space of all bounded analytic functions in $D = \{z\colon |z| < 1\}$ and let $\mathcal{B}$ be the subset of H^∞ of all functions f such that $\sup_{z\in D} |f(z)| \leqslant 1$. The Nevanlinna–Pick interpolation

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problem in D is then, given the sets $\{z_1,\dots,z_n\}$, $\{w_1,\dots,w_n\}$ in D, $z_i \neq z_j$ if $i \neq j$, to find a function $f \in \mathcal{B}$ such that

$$f(z_j) = w_j, \quad j = 1,\dots,n. \tag{1}$$

One might additionally require that the solution is of minimal norm. The existence of a solution to (1) in $\mathcal{B}$ is equivalent to the positive semi-definiteness of the Pick matrix Q_n defined by

$$Q_{ij} = \frac{1 - w_i \overline{w}_j}{1 - z_i \bar{z}_j}, \quad i, j = 1,\dots,n. \tag{2}$$

A finite Blaschke product of degree n is a rational function of the form

$$B_n(z) = c \prod_{k=1}^{n} \frac{z - \alpha_k}{1 - \bar{\alpha}_k z}, \quad |c| = 1,\ |\alpha_k| < 1. \tag{3}$$

It is clear that a finite Blaschke product of degree n always can be written in the form

$$\frac{\alpha_0 + \alpha_1 z + \cdots + \alpha_n z^n}{\bar{\alpha}_n + \bar{\alpha}_{n-1} z + \cdots + \bar{\alpha}_0 z^n}, \tag{4}$$

where $\alpha_j \in \mathbb{C}$, $j = 0,\dots,n$. The rational form (4) has constant modulus 1 on the unit circle and is a finite Blaschke product precisely when the numerator has all its zeros in D.

Our aim is to focus on three problems:

Problem 1. Let $z_1,\dots,z_n$ be distinct points such that $|z_j| = 1$, $j = 1,\dots,n_1$ ($n_1 \geqslant 1$), and $|z_j| < 1$, $j = n_1 + 1,\dots,n_1 + n_2$, and let $w_1,\dots,w_n$ be complex numbers such that $|w_j| = 1$, $j = 1,\dots,n_1$, and $|w_j| < 1$, $j = n_1+1,\dots,n_1+n_2$, where $n_1 + n_2 = n$. Suppose that the matrix Q_{n_2} formed of the data $\{z_{n_1+1},\dots,z_n\}$, $\{w_{n_1+1},\dots,w_n\}$ is positive semi-definite and of full rank. Show that there exists a finite Blaschke product B of degree at most $n-1$ with $B(z_j) = w_j$, $1 \leqslant j \leqslant n$.

Problem 2. Construct all finite Blaschke products of form (4) that solve the interpolation problem (1) for given Nevanlinna–Pick data in D with positive semi-definite matrix Q_n.

Problem 3. Construct the scaled finite Blaschke product cB_m of minimal norm on D, where B_m is of form (4) with $m \leqslant n-1$, that solves (1) for the data $\{z_1,\dots,z_n\}$ in D and $\{w_1,\dots,w_n\}$ in $\mathbb{C}$.

In the special case of Problem 1, where all $z_1,\dots,z_n$ are on the unit circle, Younis [5] gave a constructive proof that there is an interpolating finite Blaschke product which could have degree as large as n^2. In 1986, Jones and Ruscheweyh [4] showed that an interpolating Blaschke product can be chosen of degree at most $n-1$. A Blaschke product interpolation problem of this kind occurs in the design of digital filters (see [4]). We also discuss a numerical limiting procedure for obtaining an approximative solution to Problem 1 under certain assumptions.

The following result tells us exactly when (1) has a solution. For a proof see, e.g., Section I.2 in Garnett [3].

Theorem 1.1 (Pick). *The matrix Q_n is positive semi-definite ($Q_n \geqslant 0$) for given Nevanlinna–Pick data in D if and only if there is a solution to* (1) *in $\mathcal{B}$. The solution in $\mathcal{B}$ is unique if and only if the rank of Q_n is strictly smaller than n.*

(a) *The rank of Q_n is $m < n$ and $Q_n \geqslant 0$ if and only if there is a unique solution to* (1) *which is a finite Blaschke product of exact degree m.*
(b) *The rank of Q_n is n and $Q_n \geqslant 0$ if and only if there is a Blaschke product of exact degree n that solves* (1)*.*

For the solution of Problem 3 we need the following result, see Earl [2] or Garnett [3].

Theorem 1.2. *Let $z_1, \dots, z_n$ be distinct points in D and let $w_1, \dots, w_n$ be complex numbers. Among all $f \in H^\infty$, which satisfy* (1)*, there is a unique function f of minimal norm. This function has the form $f(z) = cB(z)$ where B is a Blaschke product of degree at most $n-1$.*

2. Results

Our first result is based on Theorem 1 in [4] and on Theorem 1.1 above.

Proposition 2.1. *Let $z_1, \dots, z_n$ be distinct points such that $|z_j| = 1$, $j = 1, \dots, n_1$ ($n_1 \geqslant 1$), and $|z_j| < 1$, $j = n_1+1, \dots, n_1+n_2$, and $w_1, \dots, w_n$ complex numbers such that $|w_j| = 1$, $j = 1, \dots, n_1$, and $|w_j| < 1$, $j = n_1+1, \dots, n_1+n_2$, where $n_1 + n_2 = n$. Also suppose that the matrix Q_{n_2} formed of the data $\{z_{n_1+1}, \dots, z_n\}$, $\{w_{n_1+1}, \dots, w_n\}$ is positive semi-definite and of full rank n_2. Then there exists a Blaschke product B of degree at most $n-1$ such that*

$$B(z_j) = w_j, \quad 1 \leqslant j \leqslant n.$$

Proof. Define

$$O_1(z) := \frac{z - z_n}{1 - \bar{z}_n z} \quad \text{and} \quad O_2(w) := \frac{w - w_n}{1 - \overline{w}_n w},$$

and let $z'_j := O_1(z_j)$ and $w'_j := O_2(w_j)$, $j = 1, \dots, n$. We proceed by induction on the points inside the unit circle.

First, let $n = n_1 + 1$ ($n_2 = 1,\ n_1 \geqslant 1$). Clearly the corresponding scalar matrix Q_{n_2} is positive semi-definite and of full rank. By Theorem 1 in [4] there exists a Blaschke product B_{n_1-1} of degree at most $n_1 - 1$ such that

$$B_{n_1-1}(z'_j) = w'_j/z'_j, \quad 1 \leqslant j \leqslant n_1.$$

Then $\tilde{B}(z) := zB_{n_1-1}(z)$ is a Blaschke product of degree at most n_1 satisfying $\tilde{B}(z'_j) = w'_j,\ j = 1, \ldots, n$. Defining $B(z) := (O_2^{-1} \circ \tilde{B} \circ O_1)(z)$ we obtain a Blaschke product of degree at most $n - 1$ such that $B(z_j) = w_j,\ j = 1, \ldots, n$.

Suppose that the theorem holds for $n_2 = k \geqslant 1$ and $n_1 \geqslant 1$. Given the data $\{z_1, \ldots, z_{n_1}, z_{n_1+1}, \ldots, z_{n_1+k+1}\}$, $\{w_1, \ldots, w_{n_1}, w_{n_1+1}, \ldots, w_{n_1+k+1}\}$, $n_2 = k + 1$ and $n = n_1 + n_2$, suppose that the corresponding matrix Q_{k+1} formed of the data $\{z_{n_1+1}, \ldots, z_{n_1+k+1}\}$, $\{w_{n_1+1}, \ldots, w_{n_1+k+1}\}$ is positive semi-definite and of rank $k + 1$. Then the matrix Q'_{k+1} formed of the data $\{z'_{n_1+1}, \ldots, z'_{n_1+k+1}\}$, $\{w'_{n_1+1}, \ldots, w'_{n_1+k+1}\}$ is positive semi-definite and of the same rank $k + 1$, from which follows that the matrix $\tilde{Q}_k$ formed of the data $\{z'_{n_1+1}, \ldots, z'_{n_1+k}\}$, $\{w'_{n_1+1}/z'_{n_1+1}, \ldots, w'_{n_1+k}/z'_{n_1+k}\}$ is also positive semi-definite and of rank k. These facts are proved in [3, pp. 8–9]. Thus by Theorem 1.1 (b) there exists a Blaschke product G of exact degree $k \geqslant 1$ such that $G(z'_{n_1+j}) = w'_{n_1+j}/z'_{n_1+j}$, $j = 1, \ldots, k$, and hence $|w'_{n_1+j}/z'_{n_1+j}| < 1,\ j = 1, \ldots, k$. (In order to get this result we required the assumption of full rank k to be sure that G is not a constant of modulus 1.) By the induction hypothesis there is a finite Blaschke product B_{n_1+k-1} of degree at most $n_1 + k - 1$ such that

$$B_{n_1+k-1}(z'_j) = w'_j/z'_j, \quad 1 \leqslant j \leqslant n_1 + k.$$

Then as in the first step of the induction we obtain a Blaschke product B of degree at most $n_1 + k$ satisfying $B(z_j) = w_j,\ j = 1, \ldots, n_1 + k + 1$. $\square$

Next we characterize all finite Blaschke product solutions to (1) in the full rank case.

Proposition 2.2. *Suppose that the matrix Q_n in* (2) *is positive semi-definite with* $\operatorname{rank}(Q_n) = n$ *for given Nevanlinna–Pick data in D. Then for every $\theta \in [0, 2\pi[$ there is a unique Blaschke product B_n of exact degree n that satisfies* (1) *and $B_n(1) = e^{i\theta}$. All Blaschke product solutions of exact degree n that satisfy* (1) *are obtained when θ runs through* $[0, 2\pi[$.

Proof. As in the proof of Proposition 2.1 we may without loss of generality transform the given Nevanlinna–Pick data so that z_n and w_n are moved to the origin and the corresponding matrix (2) has rank n. Let $n = 1$ and $\theta \in [0, 2\pi[$. Then $B_1(z) = e^{i\theta}z$ is the unique Blaschke product of degree 1 that satisfies (1) and $B_1(1) = e^{i\theta}$.

Suppose that Proposition 2.2 holds for $n = k - 1$. Let $n = k$ and assume that $\operatorname{rank}(Q_k) = k$ for the transformed Nevanlinna–Pick data in D. Take an arbitrary $\theta \in [0, 2\pi[$. By Theorem 1.1 (b) there is a Blaschke product solution of exact degree k to (1). Let $\tilde{B}_k$ be such a solution. Then $\tilde{B}_k = z\tilde{B}_{k-1}$, where $\tilde{B}_{k-1}$ is a Blaschke product of exact degree $k - 1$. Thus $\tilde{B}_{k-1}$ solves (1) for the data $\{z_1, \dots, z_{k-1}\}$, $\{w_1/z_1, \dots, w_{k-1}/z_{k-1}\}$, which then is Nevanlinna–Pick data with a corresponding positive semi-definite matrix Q_{k-1}. By Theorem 1.1 (b) this matrix has rank $k - 1$. Hence by the induction hypothesis there is a unique finite Blaschke product solution B_{k-1} of degree $k - 1$ to the data $\{z_1, \dots, z_{k-1}\}$, $\{w_1/z_1, \dots, w_{k-1}/z_{k-1}\}$ which also satisfies $B_{k-1}(1) = e^{i\theta}$. Thus $B_k(z) = zB_{k-1}(z)$ is the unique finite Blaschke product of degree k that satisfies (1) and $B_k(1) = e^{i\theta}$ for the data $\{z_1, \dots, z_k\}$, $\{w_1, \dots, w_k\}$.

Since a finite Blaschke product maps $z = 1$ to a point on the unit circle, all Blaschke product solutions of exact degree n can be found by allowing θ to run through the interval $[0, 2\pi[$. $\square$

Suppose that we have the points $\{z_1, \dots, z_n\}$ in D with $z_i \neq z_j$ if $i \neq j$ and the points $\{w_1, \dots, w_n\}$ in $\mathbb{C}$. We define the $n \times n$ matrices A_n, B_n and C_n by

$$A_n = \begin{pmatrix} 1 & z_1 & \dots & z_1^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & z_n & \dots & z_n^{n-1} \end{pmatrix}, \qquad B_n = \begin{pmatrix} w_1 z_1^{n-1} & \dots & w_1 z_1 & w_1 \\ \vdots & & \vdots & \vdots \\ w_n z_n^{n-1} & \dots & w_n z_n & w_n \end{pmatrix},$$
$$C_n = A_n^{-1} B_n. \tag{5}$$

C_n is well-defined because A_n is a non-singular Vandermonde matrix. We introduce the $2n \times 2n$ block matrix M_{2n} by

$$M_{2n} = \begin{pmatrix} C_n^{\mathrm{re}} & C_n^{\mathrm{im}} \\ C_n^{\mathrm{im}} & -C_n^{\mathrm{re}} \end{pmatrix}, \tag{6}$$

where C_n^{re} and C_n^{im} are the real and the imaginary parts of C_n, respectively.

Theorem 2.3. *Suppose that $\{z_1, \dots, z_n\}$ are points in D with $z_i \neq z_j$ if $i \neq j$. In* (a) *and* (b) *let $\{w_1, \dots, w_n\}$ be points in D and in* (c) *arbitrary complex numbers. Then a constructive solution to Problem* 2 *is given in* (a) *and* (b), *and to Problem* 3 *in* (c).

(a) *Suppose that Q_n is positive semi-definite and that the rank of Q_n is n. Append $z_{n+1} = 1$ and $w_{n+1} = e^{i\theta}$ for $\theta \in [0, 2\pi[$ to the data. Then the unique Blaschke product B_n of exact degree n in Proposition* 2.2 *that satisfies* (1) *and $B_n(1) = e^{i\theta}$ is determined by any real eigenvector of M_{2n+2} corresponding to the eigenvalue $k = 1$.*
(b) *Suppose that Q_n is positive semi-definite and that the rank of Q_n is $m < n$. Then by using the data $\{z_1, \dots, z_{m+1}\}$, $\{w_1, \dots, w_{m+1}\}$ in D the unique*

Blaschke product of degree m *in Theorem* 1.1 (a) *that satisfies* (1) *for* $\{z_1, \dots, z_n\}$, $\{w_1, \dots, w_n\}$ *in* D *is given by any real eigenvector of* M_{2m+2} *corresponding to the eigenvalue* $k = 1$.

(c) *The unique minimal scaled Blaschke product* $kB(z)$ *of degree* $\leqslant n-1$ *in Theorem* 1.2 *that satisfies* $kB(z_i) = w_i$, $i = 1, \dots, n$, *is given by any real eigenvector of* M_{2n} *that corresponds to the largest eigenvalue* k, $k \geqslant 0$.

In all cases, the real eigenvector gives the real and the imaginary parts of the numerator coefficients of the finite Blaschke product written in the form (4). *If the degree of the minimal scaled Blaschke product in* (c) *is strictly less than* $n-1$, *there occurs cancellation of zeros in the numerator and denominator of* (4).

Proof. Suppose we have the data $\{z_1, \dots, z_{N+1}\}$, $\{w_1, \dots, w_{N+1}\}$ as in the assumption, where $N = n$ in case (a), $N = m$ in case (b) and $N = n-1$ in (c). Then in case (a) by Proposition 2.2 we have a unique Blaschke product solution of exact degree $N = n$. In case (b) by Theorem 1.1 (a) we have a unique Blaschke product solution of exact degree $N = m$ to the data $\{z_1, \dots, z_n\}, \{w_1, \dots, w_n\}$ and this solution is also a finite Blaschke product solution to the data $\{z_1, \dots, z_{N+1}\}, \{w_1, \dots, w_{N+1}\}$ with the matrix Q_{N+1}. Therefore Q_{N+1} is positive semi-definite and the rank of Q_{N+1} is strictly less than $N+1$ since Q_{N+1} is a submatrix of the matrix Q_n which is of rank N. Hence, by Theorem 1.1, there is a unique solution in $\mathcal{B}$ which interpolates the data $\{z_1, \dots, z_{N+1}\}, \{w_1, \dots, w_{N+1}\}$. This solution is then the finite Blaschke product of exact degree m that also interpolates the data $\{z_1, \dots, z_n\}, \{w_1, \dots, w_n\}$. In case (c) by Theorem 1.2 there is an unique scaled minimal Blaschke product solution of degree $\leqslant N = n-1$. In all cases the unique solution can be obtained from the following system of equations:

$$k\frac{\alpha_0 + \alpha_1 z_j + \cdots + \alpha_N z_j^N}{\bar{\alpha}_N + \bar{\alpha}_{N-1} z_j + \cdots + \bar{\alpha}_0 z_j^N} = w_j, \quad j = 1, \dots, N+1, \tag{7}$$

where we without loss of generality can assume that $k \geqslant 0$. Then (7) has the matrix formulation

$$A_{N+1} kx = B_{N+1}\bar{x}, \quad x = (\alpha_0, \dots, \alpha_N)^T, \ k \geqslant 0.$$

This equation is equivalent to

$$C_{N+1}\bar{x} = kx, \quad k \geqslant 0. \tag{8}$$

Then (8) can be written in the form

$$\begin{pmatrix} C_{N+1}^{\mathrm{re}} & C_{N+1}^{\mathrm{im}} \\ C_{N+1}^{\mathrm{im}} & -C_{N+1}^{\mathrm{re}} \end{pmatrix} \begin{pmatrix} x^{\mathrm{re}} \\ x^{\mathrm{im}} \end{pmatrix} = k \begin{pmatrix} x^{\mathrm{re}} \\ x^{\mathrm{im}} \end{pmatrix}, \quad k \geqslant 0,$$

where x^{re} and x^{im} are the real and the imaginary parts of the vector x, respectively. (If we obtain a complex eigenvector, then because $k \geqslant 0$ and the matrix is real we may choose the real or the imaginary part of the eigenvector as a new real eigenvector corresponding to k.)

Because of the uniqueness of the desired solution in (a) and (b) the geometric multiplicity of the eigenvalue $k = 1$ is 1. In fact, suppose that the geometric multiplicity is $l \geqslant 2$. Then we could choose a basis $y_1, \ldots, y_l$ of the eigenspace such that y_1 corresponds to the desired solution $(\alpha_0, \alpha_1, \ldots, \alpha_N)^T$ with $\alpha_N \neq 0$. Let y_2 correspond to the solution $(\beta_0, \beta_1, \ldots, \beta_N)^T$. Hence $y = y_1 + \varepsilon y_2$, where $\varepsilon > 0$, is an eigenvector which corresponds to another solution $(\gamma_0^\varepsilon, \gamma_1^\varepsilon, \ldots, \gamma_N^\varepsilon)^T$ different from y_1. For ε small enough $|\varepsilon\beta_0 + \varepsilon\beta_1 z + \cdots + \varepsilon\beta_N z^N| < |\alpha_0 + \alpha_1 z + \cdots + \alpha_N z^N|$ for all z on the unit circle. From this we conclude by Rouché's theorem that $\gamma_0^\varepsilon + \gamma_1^\varepsilon z + \cdots + \gamma_N^\varepsilon z^N$ and $\alpha_0 + \alpha_1 z + \cdots + \alpha_N z^N$ have the same number N of zeros inside the unit circle. This contradicts the uniqueness. Therefore any real eigenvector of M_{2N+2} that corresponds to the eigenvalue $k = 1$ gives a solution $(\alpha_0, \ldots, \alpha_N)^T$ for which $\alpha_0 + \alpha_1 z + \cdots + \alpha_N z^N$ has all its zeros in D.

In case (c) we now show by contradiction that the minimal scaled Blaschke product corresponds to the largest non-negative eigenvalue. Indeed, let k_0 be the largest non-negative eigenvalue. Suppose that there is a non-negative eigenvalue $k_1 < k_0$ that corresponds to the minimal scaled Blaschke product $k_1 B(z)$ such that $k_1 B(z_j) = w_j$, $j = 1, \ldots, N+1$. Then k_0 corresponds to a rational function $F(z)$ of form (7) which after possible cancellation has no zeros and poles on the unit circle. Assume that the number of poles of $F(z)$ in D is equal to l, where $0 \leqslant l \leqslant N$. Then $|k_1 B(z)| < |F(z)|$ for all z with $|z| = 1$. Define $G(z) := k_1 B(z) - F(z)$. Now by Rouche's theorem for meromorphic functions [1, p. 125], we have $Z_G - P_G = Z_F - P_F$, where Z_G, Z_F (P_G, P_F) are the number of zeros (poles) of G and F in D. Since $G(z_j) = 0$, $j = 1, \ldots, N+1$, we have that $Z_G \geqslant N+1$ and $P_G = l$. Since $P_F = l$ and $F(z)$ is of form (7) it must have l zeros outside $\overline{D}$, so $Z_F \leqslant N - l$ and, therefore, $Z_F - P_F \leqslant (N-l) - l$. Consequently, $(N-l) - l \geqslant Z_G - P_G \geqslant N + 1 - l$, so $l \leqslant -1$, which gives a contradiction. Thus the minimal scaled Blaschke product always corresponds to the largest non-negative eigenvalue k_0.

Let the unique solution $k_0 B(z)$ have the numerator $p(z) := \alpha_0 + \alpha_1 z + \cdots + \alpha_{N_1} z^{N_1}$ in (7), where $N_1 \leqslant N$. Suppose that there is another real eigenvector corresponding to the largest non-negative eigenvalue k_0 which gives $k_0 \tilde{B}(z)$ with numerator $q(z) := \beta_0 + \beta_1 z + \cdots + \beta_{N_2} z^{N_2}$ in (7) with $N_2 \leqslant N$. Then $\tilde{B}(z)$ can have at most N_2 poles in D and all poles are different from z_j, $j = 1, \ldots, N+1$. Define the rational function $H(z) := B(z) - \tilde{B}(z)$ which has the same poles as $\tilde{B}(z)$ in D. The numerator of $H(z)$ is a polynomial of degree at most $2N$ which is zero in the points z_j, $j = 1, \ldots, N+1$. For $z \neq 0$ the numerator of $H(z)$ can be written in the form

$$T(z) := z^{N_2} p(z) \overline{q(1/\bar{z})} - z^{N_1} q(z) \overline{p(1/\bar{z})}.$$

If $z_j \neq 0$, then $T(z_j) = 0$ and it is easy to verify that

$$T(1/\bar{z}_j) = -\frac{\overline{T(z_j)}}{\bar{z}_j^{N_1+N_2}} = 0.$$

Then the numerator of $H(z)$ has at least $2N + 1$ zeros and, consequently, is identically zero in $\mathbb{C}$. Thus $B(z) = \tilde{B}(z)$ for all $z \in D$. Then every real eigenvector corresponding to k_0 gives the same solution of form (7). (Degeneration to degree $N_1 < N$ can occur in two ways. The numerator $\alpha_0 + \alpha_1 z + \cdots + \alpha_N z^N$ obtained from a real eigenvector may have zeros on the unit circle which then are cancelled by the corresponding poles in the denominator in (7). The numerator may also have a zero u_i of multiplicity ν_i outside $\overline{D}$ but then the numerator must have a zero $1/\bar{u}_i$ of multiplicity ν_i in D, so that the zeros u_i and $1/\bar{u}_i$ are cancelled by the corresponding poles in the denominator in (7).)

Thus we have determined the finite Blaschke product of exact degree N in the form (4) in cases (a) and (b), and the possibly degenerated finite scaled Blaschke product in case (c). □

Remark. It is well known that the Vandermonde matrix A_n can be ill-conditioned. In such a case one can compute C_n without determining the inverse of A_n. Having the equation $A_n C_n = B_n$, one observes that the kth column of C_n is given by the coefficients $x_k^{(1)}, \ldots, x_k^{(n)}$ of the polynomial $p_k(z) = x_k^{(1)} + x_k^{(2)} z + \cdots + x_k^{(n)} z^{n-1}$, which satisfies the interpolation conditions $p_k(z_j) = w_j z_j^{n-k}$, $j = 1, \ldots, n$.

Example. Suppose that we have the following data: $z_1 = 0$, $z_2 = 1/4$, $z_3 = 1/2 + i/2$, $z_4 = -i/2$ and $w_1 = -i/6$, $w_2 = -64/1015 - 102i/1015$, $w_3 = -13/25 + i/25$, $w_4 = -6/85 + 2i/17$. Then Q_4 is positive semi-definite and of rank 2. In order to solve Problem 2 we can, by Theorem 2.3 (b), use the data $\{z_1, z_2, z_3\}$, $\{w_1, w_2, w_3\}$ to find the unique Blaschke product of degree 2 that solves the interpolation problem. By numerically computing the eigenvalues and the eigenvectors of M_6 in Matlab we obtained the solution

$$B(z) = \frac{(z - 1/2)(z + i/3)}{(1 - z/2)(1 - iz/3)}.$$

For the minimal scaled Blaschke product we get the same solution when applying Theorem 2.3 (c) to the data $\{z_1, z_2, z_3, z_4\}$, $\{w_1, w_2, w_3, w_4\}$ (which was expected).

Assume now that we have the data $z_1 = 0$, $z_2 = 1/4$ and $w_1 = -i/6$, $w_2 = -64/1015 - 102i/1015$. Then Q_2 is positive semi-definite of full rank 2, so the solution of Problem 2 is not unique. By Theorem 2.3 (a) we may for any θ find a solution by appending $z_3 = 1$ and $w_3 = e^{i\theta}$ to the data. If $w_3 = 4/5 + 3i/5$, then we obtain the above solution. To find the minimal scaled Blaschke product that interpolates the data $\{z_1, z_2\}$, $\{w_1, w_2\}$ we compute the non-negative eigenvalues and the real eigenvectors of M_4. The solution is

$$kB(z) = 0.405959\frac{(-0.343583 - 0.161836i) + (0.394192 + 0.836883i)z}{(0.394192 - 0.836883i) + (-0.343583 + 0.161836i)z},$$

where $k = 0.405959$ is the norm of the minimal scaled Blaschke product.

3. A numerical limiting procedure

In the following we will discuss a numerical method for determining an interpolating finite Blaschke product of degree at most $n-1$ for data occurring in Proposition 2.1 satisfying an additional condition explained below.

Let $z_1, \ldots, z_n$ be distinct points in $\overline{D}$ such that $|z_j| = 1$, $j = 1, \ldots, n_1$ ($n_1 \geqslant 1$), and $|z_j| < 1$, $j = n_1 + 1, \ldots, n_1 + n_2$, and $w_1, \ldots, w_n$ complex numbers such that $|w_j| = 1$, $j = 1, \ldots, n_1$, and $|w_j| < 1$, $j = n_1 + 1, \ldots, n_1 + n_2$, where $n_1 + n_2 = n$. Further we assume that the matrix Q_{n_2} formed of the data $\{z_{n_1+1}, \ldots, z_n\}$, $\{w_{n_1+1}, \ldots, w_n\}$ is positive semi-definite and of full rank n_2.

For any $m \in \mathbb{N}$, $m \geqslant 2$, there exists by Theorem 1.2 a Blaschke product B_m of degree at most $n-1$, $c_m \in \mathbb{C}$, $|c_m| \geqslant 1$, such that

$$B_m\left(\left(1 - \frac{1}{m}\right)z_j\right) = \frac{w_j}{c_m}, \quad 1 \leqslant j \leqslant n,$$

where

$$B_m(z) = \prod_{k=1}^{M(m)} \frac{z - \alpha_k^{(m)}}{1 - \bar{\alpha}_k^{(m)} z}$$

with $M(m) \leqslant n-1$ and all $|\alpha_k^{(m)}| < 1$. Clearly there is a smallest integer $M \leqslant n-1$ so that we can find infinitely many $B_m(z)$ of exact degree M. Let $(B_r)_r$ be such a subsequence of $(B_m)_m$. Since $|\alpha_k^{(r)}| \leqslant 1$ for all k and r, we obtain, by going to subsequences, a subsequence $(B_l)_l$ of $(B_r)_r$ such that

$$\alpha_k^{(l)} \to \alpha_k, \quad |\alpha_k| \leqslant 1, \quad \text{for all } k = 1, \ldots, M.$$

Define the index set I to be the set of all $k \in \{1, \ldots, M\}$ such that $|\alpha_k| < 1$ and $I^c := \{1, \ldots, M\} \setminus I$. Let

$$G(z) := \prod_{k \in I^c} (-\alpha_k) \prod_{k \in I} \frac{z - \alpha_k}{1 - \bar{\alpha}_k z},$$

which is a Blaschke product of degree $|I| \leqslant M$ and $\prod_{k \in I^c}(-\alpha_k) = 1$ if $|I| = M$.

Table 1

m	$\lvert c_m\rvert$	$\max_{1\leqslant j\leqslant 5}\lvert c_m B_m(z_j)-w_j\rvert$
5	2.659435	1.764259
10	1.687972	0.709776
20	1.327671	0.332525
40	1.163320	0.164212
100	1.066006	0.066081
200	1.033220	0.033230
400	1.016677	0.016678
800	1.008357	0.008357
100000	1.000067	0.000067

Now we make the assumption that $z_j \neq \alpha_k$ for all k and all $j = 1, \dots, n_1$. (This assumption cannot hold if the Pick matrix of the data inside the unit circle is not positive semi-definite.) Then for all z_j we have

$$\lim_l B_l\left(\left(1-\frac{1}{l}\right)z_j\right) = \lim_l \prod_{k=1}^{M} \frac{(1-1/l)z_j - \alpha_k^{(l)}}{1-\bar{\alpha}_k^{(l)}(1-1/l)z_j} = G(z_j).$$

In particular, for z_1 we have $|G(z_1)| = 1$ and $|w_1| = 1$, so we conclude that

$$\lim_l c_l = \frac{w_1}{G(z_1)} = e^{i\psi}$$

for some $\psi \in [0, 2\pi[$. Thus the finite Blaschke product $B(z) := e^{i\psi}G(z)$ of degree $|I| \leqslant M \leqslant n-1$ satisfies $B(z_j) = w_j$ for $1 \leqslant j \leqslant n$.

For points z_j not satisfying the above assumption, the limit $\lim_l B_l((1-1/l)z_j)$ exists but we do not know whether it coincides with $G(z_j)$. If they always coincide, then we would have an alternative proof of Proposition 2.1.

Example. Let us consider the following data on the closed unit disc: $z_1 = 1$, $z_2 = i$, $z_3 = (-1+i)/\sqrt{2}$, $z_4 = 0$, $z_5 = 1/2$ and $w_1 = -i$, $w_2 = -1$, $w_3 = 1$, $w_4 = 1/5$, $w_5 = 1/3$. The Pick matrix corresponding to the data inside the unit circle is positive semi-definite and of full rank 2. According to Proposition 2.1 there exists a finite Blaschke product of degree at most 4 that interpolates the data. We may attempt to find an approximative solution to the problem by utilizing Theorem 2.3 (c) on the data $\{(1-1/m)z_j\}$, $\{w_j\}$, $j = 1, \dots, 5$, with a large integer m, thus obtaining the minimal scaled Blaschke product $c_m B_m$ satisfying $c_m B_m((1-1/m)z_j) = w_j$. If the assumption above that $z_j \neq \alpha_k$ for all k and all $j = 1, 2, 3$ holds and if m is chosen large enough in the subsequence, then $c_m B_m(z_j) \approx w_j$. The results for some m are listed in Table 1.

We observe in this particular example that the error in the interpolation is roughly halved when m is doubled. For $m = 100000$, $c_m B_m(z)$ is given by

$$c_m B_m(z) = c_m \prod_{k=1}^{4} \frac{z-\alpha_k^{(m)}}{1-\bar{\alpha}_k^{(m)} z},$$

where $c_m = 0.65903368 - 0.75220252i$ and the zeros of B are

$$\alpha_1^{(m)} = 0.80622573 - 0.13610365i,$$
$$\alpha_2^{(m)} = -0.60216646 + 0.76054120i,$$
$$\alpha_3^{(m)} = -0.07634867 + 0.72283345i,$$
$$\alpha_4^{(m)} = -0.33664744 - 0.08367971i.$$

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References

[1] J.B. Conway, Functions of One Complex Variables, 2nd ed., Springer-Verlag, 1978.
[2] J.P. Earl, A note on bounded interpolation in the unit disc, J. London Math. Soc. (2) 13 (1976) 419–423.
[3] J.B. Garnett, Bounded Analytic Functions, Academic Press, 1981.
[4] W.B. Jones, S. Ruscheweyh, Blaschke product interpolation and its application to the design of digital filters, Constr. Approx. 3 (1986) 405–409.
[5] R. Younis, Interpolation by a finite Blaschke product, Proc. Amer. Math. Soc. 78 (1980) 451–452.